Borromean Ring Signatures*

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Abstract

In 2002, Abe, Ohkubo, and Suzuki developed a new type of ring signature based on the discrete logarithm problem, which used a novel commitment structure to gain significant savings in size and verification time for ring signatures[AOS02].

Ring signatures are signatures using n verification keys which require knowledge of one of the corresponding secret keys. They can therefore be considered a signature of a disjunctive statement "I know x_1 OR I know x_2 OR …". We generalise their construction to handle conjunctive statements "I know one of $\{x_1, x_2, x_3, \ldots\}$ AND one of $\{x_4, x_5, x_6, \ldots\}$ AND …" and thereby gain the ability to express knowledge of any monotone boolean function of the signing keys.

This can be trivially done by use of multiple independent ring signatures; our construction saves space relative to this by sharing commitments across the individual rings.

We also describe a new way of thinking about these ring signatures, and ordinary Schnorr signatures, in terms of "causal loops" which may provide a framework for further generalisations.

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1 Random oracles and Schnorr signatures

Throughout we assume we are working in a cyclic additive group $\mathscr{G}$ for which the discrete logarithm problem is hard, and that G is some fixed known generator of the group.

1.1 Schnorr authentication

Consider the following interactive authentication protocol, developed by Claus Schnorr in 1989[Sch89], which allows the possessor of a secret discrete logarithm x of a public group element $P = xG$ to prove knowledge of x without revealing anything about it:

1. The prover chooses a random scalar k and submits kG to the verifier.
2. The verifier responds with a random challenge scalar e.
3. The prover replies with the scalar $s \leftarrow k + xe$.

The verifier does not know k, since the discrete logarithm problem is hard, but can check that s was computed honestly because $s = k + xe$ is equivalent to $sG = kG + exG$ and the verifier knows each of s, e, kG and xG.

Intuitively, this is zero knowledge because if the verifier had slipped the prover pre-knowledge of what e would be, the prover could have produced a legitimate s without knowing x at all. (Specifically, she would choose s randomly and then choose "kG" as $sG - exG$.) The transcript of the prover/verifier interactions in this case would be statistically indistinguishable from a transcript in the honest game; thus if the dishonest game revealed nothing about x (and it did not; it did not even use x!) then neither did the honest one.

Intuitively, it proves that the prover knows x, since e was chosen uniformly at random. If she could win no matter what e was, then it is a simple matter to "fork" her and give each fork different e values, say e_1 and e_2. Then the two forks would produce $s_1 = k + xe_1$ and $s_2 = k + xe_2$, which expose x as $x = (s_1 - s_2)/(e_1 - e_2)$. In other words, a verifier that can win regardless of e can be used to extract the value of x, and therefore she must have knowledge of it.

1.2 Random oracles and Fiat-Shamir

The above scheme is not *publicly verifiable*, for exactly the same reason that it is zero-knowledge. That is, a transcript of the interaction between the prover and verifier is indistinguishable from one in which they were colluding to avoid anybody knowing x. Therefore, in the honest game, the prover proves knowledge of x to the verifier, but not to anybody else.

Fortunately, since the verifier has no purpose except to produce random values in response to challenges, it has a very simple mathematical description: that of a *random function*. Random functions are mathematical functions which return independent uniformly random values on each input. "Evaluating" such a function is functionally equivalent to submitting a challenge value and receiving a new random value on each new input. Therefore random functions are often referred to as *random oracles*.

Random functions have infinite Kolmogorov complexity and cannot be instantiated in a machine, but we replace them by simple functions called *hash functions* for which there is no known way to distinguish them from random. This gives us the so-called *random oracle model*[BR93], in which we effectively have a challenger who exists in the world of Platonic forms[Ros51], and whom we refer to as a random oracle.

Random oracles have two benefits over traditional challengers: (a) the oracle's behaviour is publicly verifiable and can be seen not to be behaving dishonestly, so by "proving to the oracle" knowledge of a discrete logarithm, one actually proves this knowledge to everyone; (b) the oracle is modelled by a hash function which can be computed by anyone to recreate the transcript of interaction, so there is no need for any actual interaction.

The idea of replacing an actual challenger with a random oracle is known as the *Fiat-Shamir transform*[FS86] after the first to use the idea to turn an interactive scheme into a non-interactive publicly verifiable one. Applying the Fiat-Shamir transform to the above interactive scheme gets us the *Schnorr signature scheme* which works as follows:

1. The prover chooses a random scalar k and computes $e = H(kG)$.

2. The prover computes the scalar $s \leftarrow k + xe$ and publishes (s, e).

Since H returns a different e for different inputs, it is possible to add things to its input; for example, computing $H(m \| kG)$ for some message m. The result is a transcript containing m for which m cannot be changed without knowledge of x. This is thus a "signature of knowledge" on m.

Anyone can verify the signature's veracity by first computing $kG = sG - eP$ and checking that this is actually the committed value; *i.e.*, that $e \stackrel{?}{=} H(kG)$. The fact that $P = xG$ is used in the verification is what ties this signature to x.

2 Ring signatures

2.1 Time travel and chameleon hashes

The critical ingredient of the above authentication scheme is the *time ordering*. We observed that if the prover could see into the future and determine e before choosing kG, she could successfully authenticate without knowing x.

In the random oracle model, there is no more interaction and therefore no more time. However, the random oracle H provides its own time ordering: since for any input y it is impossible to know $H(y)$ without evaluating it (except by guessing it, which succeeds with only negligible probability), we can say that $H(y)$ is determined "after" y. We therefore preserve the idea of time being essential, though our definition of time has changed.

This definition of time admits a cryptographic trick to allow the possessor of some secret knowledge to reverse it. The way this works is to tweak our hash function H to make it a *chameleon hash*[KR97]. A chameleon hash, rather than taking just an input e, also takes as input a random value s. A possessor of some secret "trapdoor" information will be able to tweak s so as to change e without changing the hash output, while ordinary mortals remain bound by the laws of time: once the hash of e is computed, e is in the past and cannot be changed any more than you can choose not to have read this sentence.

We define a chameleon hash from an ordinary hash H as follows. Here G and P are generators of the group; the trapdoor information is x such that $P = xG$. Our hash function is

$$H'(m,e,s) = H(m\|sG - eP)$$

(This is actually a "half-chameleon hash": someone with trapdoor information can change the value of e by changing s appropriately[1], but nobody can change m without changing the output of the hash function.)

We notice that if $s = k + xe$, which is the value s from a Schnorr signature, then $e = H'(m,e,s) = H(m\|kG)$ can be computed without foreknowledge of e. We can therefore describe Schnorr signatures as a pair (s,e) where e is both the input and output of a chameleon hash and s is its random input. Since the output of the hash is random and independent of its input, forcing the input to be equal to the output requires trapdoor information, and is thus a proof that the signer has this information. The result is called a *signature of knowledge*.

In other words, we can think of Schnorr signatures as working as follows: to produce a Schnorr signature without knowing the secret key x, one must predict the output of random oracle H, effectively "travelling backward in time". The signature is structured to essentially contain a hash of itself, creating a causal loop and forcing signers to know the trapdoor information.

In the next section we will generalise these causal loops and see that they are a useful abstraction.

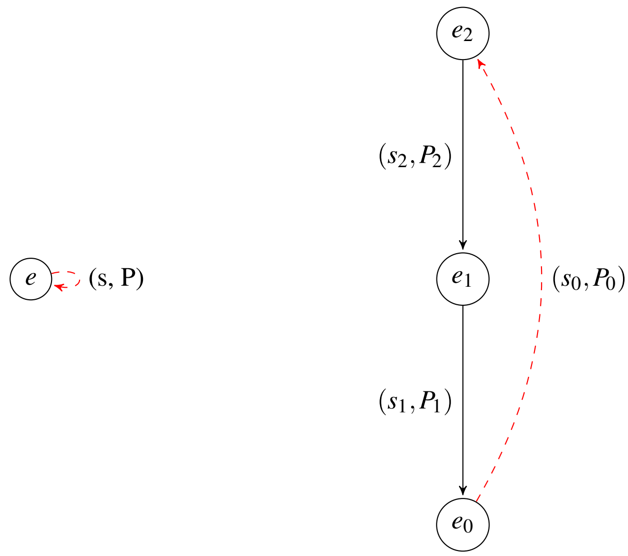
2.2 Abe-Ohkubo-Suzuki ring signatures

Ring signatures are a variant of digital signatures in which the verification key is replaced by a set, or *ring*, of verification keys. Each verification key has a corresponding secret key, and only one is required to produce a ring signature. However, all of the verification keys play the same role in verifying the signature, so the specific signing key used remains secret.

Ring signatures were introduced by Rivest, Shamir, and Tauman[RST01] in 2001. Their suggested use-case was whistleblowing: a well-connected signer could construct a ring with the verification keys of other well-connected parties, then sign a message blowing the lid off some conspiracy. Verifiers would see that somebody well-connected had signed off on the leak, giving it veracity, but they would not know who specifically had done it, preventing nasty personal consequences.

In 2002, Abe, Ohkubo, and Suzuki developed a new type of ring signature based on the discrete logarithm problem[AOS02], which used causal loops (though they did not use the term) to obtain the ring property. This use of causal loops gave a significant (50%) reduction in the size of the signatures compared to earlier ring signatures. It is described as follows.

[1] Specifically, if $h = H'(m,e,s)$, then to change e to e' one sets $s' \leftarrow s + (e - e')x$, so $s'G - e'P = sG - eP$ and thus $H'(m,e',s') = H(m,e,s)$.



Schnorr Signature (1 of 1) AOS Signatures (1 of 3)

Figure 1: Schnorr as a special-cased AOS signature

In this scheme, rather than having $kG = sG + exG$ committed to by e, we have a set of hashes $\{e_i\}_{i=0}^{n-1}$, each committing to $k_iG = s_iG + e_iP_i$, where the indices are considered modulo n. Since each e_i commits to the s_{i-1} before it, it is easy to produce values for k_iG — just choose s_i at random and compute the resulting k_iG and e_i! The signer can do this starting with any index j: first compute e_j as the hash of some random k_jG, then using this compute (s_{j+1}, e_{j+1}), then (s_{j+2}, e_{j+2}), and so on until reaching (s_j, e_j).

However, when computing (s_j, e_j), the signer finds that e_j has already been determined (this was the first step). He must somehow compute s_j to fit. Just like with the ordinary Schnorr signature, to do this requires "going back in time" to choose an s_j value compatible with e_j, and just like with the ordinary Schnorr signature, this can be done as long as the signer knows the discrete logarithm x_j of P_j: set $s_j = k_j - x_je_j$. The final signature is $\sigma = \{e_0, s_0, s_1, \ldots, s_{n-1}\}$.

Since s_j contains the random value k_j, it appears uniformly random, just like every other s_i, and from the perspective of the verifier, there is nothing special about the index j. The result is a signature which requires many verification keys to verify, only one of whose secret key is required to produce. This is a ring signature!

This has been very algebraic. We can extract the structure of the commitments to see what is "really" going on with these ring signatures. We do this by drawing them as directed graphs. The vertices of the graph are labelled by the chameleon hash output; the incoming edges by their blinded input. (As in the Schnorr case, the blinding either forces the value of the edge's source vertex or requires some secret knowledge; the difference is that in the Schnorr case there is only one edge whose source and target vertices are the same.)

In other words: as before, we have a notion of time which mandates nothing more than that a hash e can only be known *after* its input is decided. And as before, we can use chameleon hashes to override this time-ordering (only) for people in possession of trapdoor information. We can draw this time-ordering as a directed graph, and make two observations:

1. To form a cycle, it is necessary and sufficient to "go back in time" for one commitment; that is, only one secret key is required.

2. From the perspective of a verifier, a reversed-time commitment is indistinguishable from an ordinary one. Further, he can only see the graph structure: the colouring and up/down distinction we have used in our illustrations is invisible.

These two observations together describe how the Abe-Ohkubo-Suzuki ring signature works: it is simply a cycle of chameleon-hash commitments[2].

3 Borromean ring signatures

With this background in place, the concept of Borromean ring signatures can be described as a straightforward generalisation. Where ordinary ring signatures take a set of verification keys $\{v_i\}_{i=1}^n$ and describe a signature signed with s_1 OR s_2 OR $\cdots$ OR s_n, where s_i is the secret key corresponding to v_i, Borromean ring signatures can describe signatures signed with arbitrary functions of the signing keys.

More formally, let $\mathscr{V}$ be some set of verification keys, and f be a function which maps finite subsets of $\mathscr{V}$ to $\{0,1\}$. We call f an *admissibility function*; then an *admissible set* V of verification keys is one for which $f(V) = 1$.

A Borromean ring signature σ is a signature on a message m with a set $\mathscr{V}$ of verification keys and admissibility function f which satisfy the following:

1. σ can be produced only by parties who together know all the secret keys to an admissible set V of verification keys.

2. Given only σ, $\mathscr{V}$, and m, it is statistically indistinguishable which admissible set V was used.

3.1 Monotone functions

We observe that if V is an admissible set, without loss we may also assume that any superset V' is also admissible. The reason is that if V is admissible, then parties holding the signing keys to V' are able to produce valid signatures by simply ignoring the keys in V' that are not in V. We can therefore put the following restriction on f: if $f(V) = 1$, then for all $V' \supseteq V$, $f(V') = 1$. Functions which satisfy this restriction are called *monotone functions*.

[2] As before, we use half-chameleon hashes; each e_i commits to the previous e_{i-1} in a time-reversible way but commits to the message and verification key set in an irreversible way, since these should not be tamperable.

3.2 AND and OR

It can be shown[BL88] that monotone functions are exactly those that can be represented by circuits in disjunctive normal form which have no negations; *i.e.*, circuits of the form

$$\bigwedge_i \left(\bigvee_j a_{i,j} \right)$$

where $\bigwedge$ or AND outputs 1 iff all its inputs are 1; $\bigvee$ or OR outputs 0 iff all its inputs are 0.

We observe that Abe-Ohkubo-Suzuki signatures can be described as Borromean ring signatures whose admissibility function f consists of a single OR gate. We observe more generally that we can construct OR gates out of cycles, as described in an earlier section.

To take the conjunction of these OR gates, it is sufficient to simply create disjoint graphs, which would each correspond to a single AOS signature. However, we can do better than this by creating a new graph structure corresponding to the AND gate. Specifically, if we create a vertex e with multiple outgoing edges (meaning multiple s values which individually either require secret knowledge or force the value of e), then we have constructed an AND without requiring multiple graphs.

It is easy to see how a signature can be structured so that its graph has vertices with multiple outgoing edges: each edge i from a vertex labelled (e,x) is labelled by a s_i value which is either random (for forward-time edges) or equal to $s_i = k_i - xe$ (for backward-time edges). However, to force backward-time edges, we need to have every path be part of a cycle. This forces us to also have vertices with multiple *incoming* edges (see vertex e_0 in Figure 2). This means we need to change our commitment structure.

Recall that for a vertex $(e,\cdot)$ with a single incoming edge s (whose source vertex is (e',x')), the value e is computed by chameleon hash as

$$e = H(m \| sG - e'x'G)$$

If we have multiple incoming edges $\{s_i\}_{i=1}^n$ whose source vertices are $\{(e_i,x_i)\}_{i=1}^n$, the generalisation is obvious:

$$e = H(m \| s_1G - e_1x_1G \| \cdots \| s_nG - e_nx_nG)$$

The result is a "multiply-chameleon hash": each x_i is a piece of trapdoor information which can be used independently to make its corresponding edge s_i go backward in time. This is exactly the structure we would expect from a vertex in our graph with multiple incoming arrows.

With these pieces in place, we are able to draw a graph describing a Borromean signature:

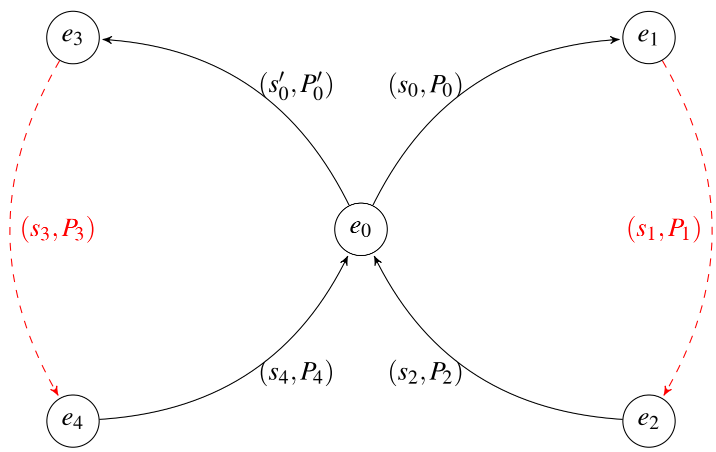


Figure 2: A Borromean ring signature for $(P_0|P_1|P_2)\&(P'_0|P_3|P_4)$

Though it would result in a more crowded picture, it is clear how this scales to more than two rings; the resulting signature is one that can only be created by knowing the discrete logarithms of one of $\{P_0, P_1, P_2\}$ *and* $\{P'_0, P_3, P_4\}$, and we saved one commitment versus having separate rings.

In general, if we take the conjunction of n rings, we save $(n-1)$ commitments versus using separate ring signatures.

3.3 Concrete algorithm

The above has given a description of Borromean ring signatures in terms of graph structures, and technically has all the information required to implement them. However, the devil is in the details, and it is not obvious that it is actually possible to compute these signatures. We therefore lay out the actual signing and verification algorithms.

3.3.1 Signing

Suppose a signer has a collection of verification keys $P_{i,j}$ for $0 \le i \le n-1$, $0 \le j \le m_i - 1$, and wants to create a signature of knowledge of the n keys $\{P_{i,j_i^*}\}_{i=0}^{n-1}$ where the j_i^*'s are some fixed and unknown (to a verifier) indices. Denote the secret key to P_{i,j_i^*} by x_i. He acts as follows:

1. Compute M as the hash of the message to be signed and the set of verification keys.
2. For each $0 \le i \le n-1$:
   (a) Choose a scalar k_i uniformly at random.
   (b) Set $e_{i,j_i^*+1} = H(M \| k_i G \| i \| j_i^*)$.
   (c) For j such that $j_i^* \le j < m_i - 1$ choose $s_{i,j}$ at random and compute

$$e_{i,j+1} = H(M \| s_{i,j} G - e_{i,j} P_{i,j} \| i \| j).$$

3. Choose s_{i,n_j} for each i at random and set

$$e_0 = H(s_{i,m_j}G - e_{i,m_j}P_{i,j} \| \cdots \| s_{n,m_j}G - e_{n,m_j}P_{i,j})$$

   That is, e_0 commits to several s-values, one from each ring.

4. For each $0 \le i \le n-1$:

   (a) For j such that $0 \le j < j_i^* - 1$ choose $s_{i,j}$ at random and compute

$$e_{i,j+1} = H(M \| s_{i,j}G - e_{i,j}P_{i,j} \| i \| j).$$

   where "$e_{i,0}$" means e_0. Note that this calculation is identical to the one in Step 2c.

   (b) Set $s_{i,j_i^*} = k_i + x_i e_{i,j_{i-1}^*}$.

The resulting signature on m consists of

$$\sigma = \{e_0, s_{i,j} : 0 \le i \le n, 0 \le j \le m_i\}$$

3.3.2 Verification

Since verification does not depend on which specific keys are known, it avoids the "two-phase" structure of signing and is therefore much simpler.

We assume we have a message m, a collection $\{P_{i,j}\}$ of verification keys whose indices range as before, and a signature σ whose notation is the same as before. The verifier acts as follows:

1. Compute M as the hash of the message to be signed and the set of verification keys.

2. For each $0 \le i \le n-1$, for each $0 \le j \le m_j - 1$, compute $R_{i,j+1} = s_{i,j}G + e_{i,j}P_{i,j}$ and $e_{i,j+1} = H(M \| R_{i,j+1} \| i \| j+1)$. (As before, we always take $e_{i,0}$ to be e_0.)

3. Compute $e_0' = H(R_{0,m_0} \| \cdots \| R_{n,m_n})$ and return 1 iff $e_0' \stackrel{?}{=} e_0$.

3.4 Efficiency comparison

Finally, we compare our scheme to existing ring signatures. We consider signing with N verification keys across n rings.

Scheme	**Size of signature**
n original CDS ring signatures[CDS94] (for example, used by Monero)	$2N$
n AOS ring signatures	$N+n$
1 Borromean ring signature	$N+1$

Here "size" is measured in field elements or hashes, which are 32 bytes for 128-bit security.

4 Open Problems

In the above, we developed signatures which are proportionate in size to the number of $a_{i,j}$'s in the disjunctive normal form expression.

$$\bigwedge_i \left(\bigvee_j a_{i,j} \right)$$

By avoiding disjunctive normal form it is possible to represent these circuits in much less space; however, it is unclear how the signatures corresponding to such circuits should be structured.

Similarly, by using threshold gates rather than only AND and OR, a space savings can be obtained for many monotone functions; it is also open how to translate this to our framework.

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